JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 59, 342–359 (1977)

Termination of Families of Periodic Solutions

RUSSELL RIMMER

Department of Pure Mathematics, La Trobe University, Bundoora, Australia 3083

Submitted by Jane Cronin

1. INTRODUCTION

Early this century E. Strömgren calculated numerically many groups, or families, of periodic solutions of the restricted three-body problem. His results lead him to announce the *Principle of Natural Termination* according to which a family of periodic solutions can only terminate in one of four ways: by the period, the dimensions of the orbits, or the energy approaching infinity or the orbits approaching a critical point.

Wintner [10] claimed to have proved the principle of natural termination, not only for the restricted three-body problem, but for any analytic system of differential equations having an integral. Wintner [9, p. 442] says that Birkhoff [3, p. 707] gives a shorter proof which is essentially the same as his. However, in his proof, Birkhoff [3, p. 707] applies the implicit function theorem to a system of equations without checking that its Jacobian is not zero. The principle of natural termination has recently been referred to by Szebehly [8, p. 487], who believes it to be true, at least for the restricted three-body problem. A more recent reference to Winter's theorem is made by Siegel and Moser [7, p. 147].

In Section 4 we obtain a counterexample to Winter's theorem (by constructing a family of periodic solutions to a simple integrable system which terminates in a way other than those given by Wintner). The main purpose of this paper is to formulate and prove a modified version of the principle of natural termination which allows a family to terminate in two additional ways not envisaged by Wintner. This result is stated in Section 3.

Because of the previous mistakes in this area we spend some time in Section 2 carefully formulating the ideas of a family of periodic solutions and extensions of such a family.

2. NOTATION AND DEFINITIONS

Notation. Let $f\colon U \subseteq \mathbb{R}^n \to F$, where U is open and F is a Banach space. If f is Fréchet differentiable at $u \in U$ then we denote its derivative at u by $Df(u)$. If $g\colon I \subseteq \mathbb{R} \to F$, where I is an open interval, we define $g'\colon I \to F$ by

$$g'(t) - Dg(t)(1).$$

ISSN 0022-247X

Let $\mathscr{C}^r(U, \mathbb{R}^n)$ be the vector space of r times continuously differentiable functions mapping $U \subseteq \mathbb{R}^m$ into $\mathbb{R}^n$ for $1 \leqslant r \leqslant \infty$. If f is r times continuously differentiable and its domain and range are clear from the context we say "f is $\mathscr{C}^r$." The subspace of bounded mappings in $\mathscr{C}^r(U, \mathbb{R}^n)$ with the supremum norm is denoted by $\mathscr{B}^r(U, \mathbb{R}^n)$. The supremum norm is written as $\| \ \|$, and, for any positive integer m, $| \ |$ denotes the norm on $\mathbb{R}^m$. If $a \in \mathbb{R}^m$ we write it in component form as $(a_1, a_2, \ldots, a_m)$.

The differential equation. We are concerned with the autonomous differential equation

$$\phi' = X \circ \phi, \tag{1}$$

where X: $\mathrm{cl}(U) \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is $\mathscr{C}^2$ on $\mathrm{cl}(U)$, the closure of U. For each $a \in U$ there is an open interval I and a unique solution ϕ: $I \to U$ of (1) with $\phi(0) = a$ (see [1, p. 40]).

A solution ϕ: $\mathbb{R} \to U$ is *periodic* if there is a positive real number ω such that for some t, hence for all t, $\phi(\omega + t) = \phi(t)$. The smallest such period is called the *prime period* (see [1, p. 156]). It is convenient to restrict the domain of such a solution ϕ to an interval L of length at least ω.

For each $a \in U$ there is a $\mathscr{C}^2$ mapping F: $V \times J \to U$ such that $V \times J$ is open in $U \times \mathbb{R}$, J is an interval containing zero, the mapping $t \mapsto F(a, t) = F_a(t)$ satisfies (1), and $F_a(0) = a$ (see [1, p. 38]). The map F is called a *flow box* for (1) at a. If a is the initial value for a periodic solution, we can choose the interval J to be arbitrarily large at the expense of taking V to be a sufficiently small neighborhood of the image of the solution.

The *characteristic multipliers* of X at a periodic solution $\phi : L \to U$ with prime period ω are the eigenvalues of the matrix of the linear map $DF^\omega(a)$: $\mathbb{R}^n \to \mathbb{R}^n$, where F^ω: $u \to F(u, \omega) = F^\omega(u)$ and $a = \phi(0)$.

An *integral* η: $U \to \mathbb{R}$ for (1) is a $\mathscr{C}^1$ function such that for each solution ϕ: $I \to U$ of (1) and for every $t \in I$

$$\eta(\phi(t)) = \eta(\phi(0)).$$

The integral is said to be *nonstationary* at $a \in U$ if $D\eta(a) \neq 0$.

Families and extensions.

DEFINITION. A function Φ: $X \to Y$ is a *diffeomorphism* of sets X and Y contained in Banach spaces if and only if Φ carries X injectively onto Y and both Φ and Φ^{-1} can be extended to differentiable functions on open sets U and V containing X and Y, respectively (see [6, p. 1]).

DEFINITION. A *family* of periodic solutions to (1) is a diffeomorphism Φ: $I \to \mathscr{B}^r(L, \mathbb{R}^n)$ where I is a bounded, open interval, each member of $\Phi[I]$ is a periodic solution of (1), and, furthermore, the function $\tau\colon I \to \mathbb{R}^+$, which assigns to γ the prime period of $\Phi(\gamma)$, is continuous.

This is a narrow definition of family. It excludes the following example: Let $\phi\colon L \to U$ be a periodic solution of (1) and define $\Phi\colon (0, 1) \to \mathscr{B}^1(L, \mathbb{R}^n)$ by $\Phi(\gamma) = \phi$ for all $\gamma \in (0, 1)$. Here Φ is a constant map, and hence is *not* a diffeomorphism.

DEFINITION. A *right extension* of a family $\Phi\colon I \to \mathscr{B}^r(L, \mathbb{R}^n)$ is a family $\Phi^*\colon J \to \mathscr{B}^r(L, \mathbb{R}^n)$ such that

(i) J is an open bounded interval and J contains I together with its right end point, and

(ii) $\Phi^* \mid I = \Phi$.

A *left extension* is defined similarly.

EXAMPLE. We give an example of a family which cannot be extended on the left but may be reparametrized to make it extendable. Consider the system of differential equations

$$\phi_1'' = -\phi_1,$$

$$\phi_2'' = -\phi_2.$$

Let $\Phi\colon (0, 1) \to \mathscr{B}^1([0, 2\pi], \mathbb{R}^2)$ be the family of periodic solutions to this system defined by putting

$$\Phi(\gamma) = (\gamma \sin, \gamma^{1/2} \sin)$$

for each $\gamma \in (0, 1)$. No extension of Φ can be differentiable at $\gamma = 0$; hence the family γ has no left extension. However, let us compose Φ with the squaring function $\iota^2\colon (0, 1) \to (0, 1)$ so that

$$(\Phi \circ \iota^2)(\gamma) = (\gamma^2 \sin, \gamma \sin).$$

From the definition this function is Fréchet differentiable. The map $\Psi = \Phi \circ \iota^2$ has inverse $\Psi^{-1}\colon \Psi[(0, 1)] \to (0, 1)$ defined by putting

$$\Psi^{-1}(f_1, f_2) = f_2(\tfrac{1}{2}\pi)$$

for $(f_1, f_2) \in \Psi[(0, 1)] \subset \mathscr{B}^1([0, 2\pi], \mathbb{R}^2)$. The same formula allows us to extend the map Ψ^{-1} to a differentiable map defined on an open neighborhood of the set $\Psi[(0, 1)]$. Therefore Ψ is a diffeomorphism, hence a family (in our sense) of periodic solutions.

The family of periodic solutions $\Psi = \Phi \circ \iota^2$, unlike the family Φ, clearly has a left extension.

This example suggests that we should allow for the possibility of "reparametrizing" before trying to extend a family. To this end we make the following definition.

DEFINITION. Let $\Phi: I \to \mathscr{B}^r(L, \mathbb{R}^n)$ be a family of periodic solutions to (1). A *reparametrization* of Φ is a family $\Phi \circ g: J \to \mathscr{B}^r(L, \mathbb{R}^n)$, where $g: J \to I$ such that $g' > 0$, g maps J onto I, and J is a bounded interval.

Any periodic solution $\phi \in \mathscr{B}^r(L, \mathbb{R}^n)$ of (1) with period ω can be embedded in a family as follows. Since (1) is autonomous the function $t \mapsto \phi(t + \gamma)$ is also a solution of (1) for each real γ. Hence if I is an interval of length at most ω the map $\Phi: I \to \mathscr{B}^r(L, \mathbb{R}^n)$ with

$$\Phi(\gamma) = (t \mapsto \phi(t + \gamma))$$

is a family of periodic solutions of (1). We call Φ a *trivial family*.

DEFINITION. A family of periodic solutions to (1), $\Phi: I \to \mathscr{B}^r(L, \mathbb{R}^n)$, is said to be *proper* if each reparametrization of it is nowhere tangent to a trivial family.

DEFINITION. A proper extension of a family of periodic solutions $\Phi: I \to \mathscr{B}^r(L, \mathbb{R}^n)$ to (1) is a proper family of periodic solutions $\Phi^* : J \to \mathscr{B}^r(L, \mathbb{R}^n)$ to (1) which is either a right or a left extension of Φ.

3. EXTENDING A FAMILY OF PERIODIC SOLUTIONS

Let Φ: $(a, b) \to \mathscr{B}^r(L, \mathbb{R}^n)$ be a family of periodic solutions to (1). We define functions τ, s: $(a, b) \to \mathbb{R}^+$ by

$$\tau(\gamma) = \text{prime period of the periodic solution } \Phi(\gamma)$$

and

$$s(\gamma) = \sup\{|\Phi(\gamma)(t)| : t \in L\}.$$

PROPOSITION 3.1. *Let* Φ: $(a, b) \to \mathscr{B}^r(L, \mathbb{R}^n)$ *be a family of periodic solutions to* (1). *Suppose that*

(A) τ, s: $(a, b) \to \mathbb{R}^+$ *are bounded functions.*

Then there is a sequence $\{\gamma_n\}_{n=1}^{\infty}$ *in* (a, b) *converging to* b *and a solution* ϕ_0 *of* (1) *which is a periodic or an equilibrium solution to* (1) *such that*

$$\Phi(\gamma_n) \to \phi_0 \text{ as } n \to \infty.$$

Possibly ϕ_0 *assumes values on the boundary of the open set* U.

THEOREM 3.2 (Extension Theorem). *Suppose that* (1) *has an integral* η *and let* $\Phi\colon (a, b) \to \mathscr{B}^r(L, \mathbb{R}^n)$ *be a family of periodic solutions to* (1). *Suppose that condition* (A) *of Proposition* 3.1 *is satisfied and so assume that* L *is closed and bounded. In addition suppose that the solution* ϕ_0 *given by the proposition satisfies*

(B) $\phi_0[L] \subseteq U$ *and* ϕ_0 *is not an equilibrium solution,*

(C) $\eta\colon U \to \mathbb{R}$ *is nonstationary on* $\phi_0[L]$,

(D) *the characteristic multiplier* 1 *of* (1) *at* ϕ_0 *has multiplicity at most* 2,

(E) *the prime period of* ϕ_0 *is* $\lim_{\gamma \to b^-} \tau(\gamma)$.

If the family Φ *is proper then either there is a reparametrization of* Φ *which has a proper right extension or* $\Phi[(a, b)]$ *is contained in a set of periodic solutions which is diffeomorphic to* S^1, *the unit circle.*

Remarks. 1. The theorem is still valid if we replace condition (D) by the condition that the characteristic multiplier 1 of (1) at ϕ_0 has multiplicity greater than 2 but the matrix of $DF^{\omega}(a) - \iota$ has rank $n - 1$, where F is the flow box of (1) at a, $\iota\colon \mathbb{R}^n \to \mathbb{R}^n$ is the identity function, and ω is the prime period of ϕ_0.

2. Intuitively: condition (E) fails if $\Phi[(a, b)]$ consists of solutions which make p circuits before closing and the limiting solution closes after one circuit. Wintner [11, pp. 611–616] has obtained some results concerning characteristic multipliers in this situation.

4. COUNTEREXAMPLE TO WINTNER'S THEOREM

Wintner [10, Sect. 5] proves a theorem which is analogous to our Theorem 3.2. His context is slightly more general in that he assumes the existence of an "invariant relation" instead of an integral. The essential difference between his theorem and ours is that he omits hypotheses (C), (D), and (E). We now give a counterexample to Wintner's theorem.

To this end we construct a family of periodic solutions to a Hamiltonian system which has no right extension and yet satisfies the hypotheses of Wintner's theorem, namely, that the prime periods, the energy, and the dimensions of the orbits are bounded, and the family does not approach a critical point. Consider the Hamiltonian system

$$x_1'' = -4x_1,$$
$$x_2'' = -x_2,$$

with the Hamiltonian integral $H(x_1, x_2, p_1, p_2) = x_1^2 + x_2^2 + p_1^2 + p_2^2/4$, where x_1, x_2, p_1, p_2 are the canonical coordinates. Consider the family of periodic solutions given by Φ: $(0, 1) \to \mathscr{B}^r([0, 2\pi], \mathbb{R}^4)$, where for each $\gamma \in (0, 1)$ the first two component functions are

$$\Phi_1(\gamma) = \gamma \cos \circ (2\iota + (1/(\gamma - 1)))$$

and

$$\Phi_2(\gamma) = \gamma \sin,$$

where ι: $\mathbb{R} \to \mathbb{R}$ is the identity function.

If this family had a continuous extension on the right the same would be true of the function $\gamma \mapsto \cos(1/\gamma - 1))$ for $\gamma \in (0, 1)$.

For the same reason, there can be no reparametrization of Φ, which has a continuous right extension.

5. An Embedding Theorem

Here we prove a local theorem which plays a key rôle in the proof of our extension theorem. This theorem includes as a special case the theorem given in [1, p. 178] for Hamiltonian systems.

Throughout this section we let $\phi_0 : L \to \mathbb{R}^n$ be a fixed periodic solution of (1) with period ω such that $\phi_0(0) = a$. Let $(\phi_0')_j$ be the jth component of the derivative $\phi_0' : L \to \mathbb{R}^n$.

Lemma 5.1. *If $(\phi_0')_j$ is nonzero at a then there exists a neighborhood N of a in $\mathbb{R}^n$ such that every periodic solution of* (1) *which has a value in N, passes through the $n - 1$ dimensional hyperplane $u_j = a_j$ where u_j is the jth coordinate of $\mathbb{R}^n$.*

Lemma 5.2. *Let Φ: $(a, b) \to \mathscr{B}^r(L, \mathbb{R}^n)$ be a diffeomorphism whose image consists of periodic solutions to* (1) *such that*

$$\Phi(\gamma) \to \phi_0 \text{ as } \gamma \to b.$$

There is no sequence $\{\gamma_n\}_{n=1}^{\infty}$ converging to b for which there exists $\omega^* \in \mathbb{R}$ such that

$$\tau(\gamma_n) < \omega^* < \omega,$$

where $\tau(\gamma_n)$ = prime period of the periodic solution $\Phi(\gamma_n)$ and ω is the prime period of ϕ_0.

For $a \in U$ define Λ: $V \times J \to \mathbb{R}^n$ by putting

$$\Lambda(u, \sigma) = F(u, \sigma) - u,$$

where F: $V \times J \to U$ is the flow box of (1) at a.

Let $[N]$ denote the matrix of the linear map N: $\mathbb{R}^m \to \mathbb{R}^p$ with respect to the usual bases of $\mathbb{R}^m$ and $\mathbb{R}^p$.

LEMMA 5.3. *If* (1) *has an integral* $\eta\colon U \to \mathbb{R}$ *then*

$$[D\eta(a)][D_1A(a,\omega)] = 0$$

where D_1A *denotes the first partial derivative of* A,

$$[D_1A(a,\omega)]\, X(a) = 0, \tag{3}$$

$$[D\eta(a)]\, X(a) = 0. \tag{4}$$

It is easy to prove Lemma 5.1 by using the continuity of solutions with respect to their initial conditions. Lemma 5.2 has a simple proof by contradiction using the theorem that continuous functions preserve convergent sequences. The results of Lemma 5.3 are standard. Their proofs can be found in [7, pp. 143–146].

LEMMA 5.4. *Let* $\mathscr{I} \subseteq \mathbb{R}^n$ *be a set of initial conditions for a set* $\mathscr{P} \subset \mathscr{B}^r(L, \mathbb{R}^n)$ *of periodic solutions of the differential equation* (1). *If there is a diffeomorphism* $f\colon I \to \mathscr{I}$, *where* I *is a bounded open interval, and if the prime period function* $\tau\colon I \to \mathbb{R}^+$ *is continuous and bounded on* I, *then there exists a family of periodic solutions to* (1), $\Psi\colon I \to \mathscr{P}$, *mapping* I *onto* $\mathscr{P}$ *such that for each* $\gamma \in I$

$$\Psi(\gamma) = \textit{periodic solution of } (1) \textit{ with initial condition } f(\gamma).$$

We prove this lemma later.

THEOREM 5.5. (Embedding Theorem). *Let* $\phi_0 : L \to U$ *be a periodic solution to* (1) *with prime period* $\omega > 0$. *Suppose that the integral* $\eta\colon U \to \mathbb{R}$ *is nonstationary on* $\phi_0[L]$. *Further suppose that* L *is a closed, bounded interval.*

If the characteristic multiplier 1 *of* (1) *at* ϕ_0 *has multiplicity at most* 2 *then there is a proper family of periodic solutions* $\Psi\colon (c-d, c+d) \to \mathscr{P} \subseteq \mathscr{B}^r(L, \mathbb{R}^n)$ *to* (1) *such that*

(a) $\Psi(c) = \phi_0$,

(b) *there is* $\rho \in \mathbb{R}^+$ *such that* $\mathscr{P}$ *contains all periodic solutions of* (1) *which lie in the open ball* $B_\rho(\phi_0) \subset \mathscr{B}^r(L, \mathbb{R}^n)$ *and have prime period in the interval* $(\omega - \rho, \omega + \rho)$.

Remark. If the multiplicity of the characteristic multiplier 1 of (1) at ϕ_0 exceeds 2 and the rank of the matrix of $DF^\omega(a) - \iota$ is $n - 1$, the conclusion is still true. This has been proved by Siegel and Moser [7, pp. 143–146].

Proof of Theorem 5.5. Our proof uses the "continuation method" of Poincaré as expounded by Siegel and Moser [7, pp. 145–148]. By reducing a Jacobian matrix to Jordan normal form, we are able, however, to deal completely with the case where two characteristic multipliers are 1. The case where 1 characteristic multiplier is 1 is treated in [7, pp. 143–147].

Finding a map f from an interval onto a set of initial conditions for periodic solutions. Recall that we defined $\Lambda: V \times J \subseteq \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\Lambda(u, \sigma) = F(u, \sigma) - u,$$

where $F: V \times J \to \mathbb{R}^n$ is the flow box of (1) at $\phi_0(0)$. Thus $\Lambda(u, \sigma) = 0$ if and only if u is the initial condition for a periodic solution ϕ to (1) with period σ. By a preliminary transformation we reduce the matrix of $D_1\Lambda(a, \omega)$ to its Jordan normal form, where $a = \phi_0(0)$. If (1) has characteristic multiplier 1 of multiplicity 2 the Jordan normal form is

$$M = \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & 0 \\ 0 & M' \end{pmatrix} \qquad \text{or} \qquad M = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} & 0 \\ 0 & M' \end{pmatrix},$$

where M' is a nonsingular, $(n - 2)$ square matrix. We now follow the procedure of Siegel and Moser [7, p. 147]. As the proof is similar in the two cases we give the proof only in the second case.

From Lemma 5.3 Eqs. (2) and (3), and the normal form for $M = [D_1\Lambda(a, \omega)]$,

$$\begin{aligned} X_i(a) &= 0, \\ D_i\eta(a) &= 0, \end{aligned} \qquad i = 3, 4, \ldots, n,$$

and so, from Lemma 5.3, Eq. (4),

$$X_1(a)\, D_1\eta(a) + X_2(a)\, D_2\eta(a) = 0.$$

As a is not a critical point of X then $X(a)$ is nonzero, and as η is nonstationary then $D\eta(a) \neq 0$, so we can assume that either $X_1(a)$ and $D_2\eta(a)$ or $X_2(a)$ and $D_1\eta(a)$ are nonzero.

We consider the proof only in the case

$$X_1(a) \neq 0 \qquad \text{and} \qquad D_2\eta(a) \neq 0,$$

as the proof in the other case is similar. Let $A \subset \mathbb{R}^{n-1}$ be the set of points $\{(u_2, \ldots, u_n) \in \mathbb{R}^{n-1} \mid (a_1, u_2, \ldots, u_n) \in V\}$. Consider the system of equations

$$\Lambda^*(\sigma, u_2, \ldots, u_n, \delta) = 0, \tag{5}$$

where $\Lambda^* : J \times A \times \mathbb{R} \to \mathbb{R}^n$ is the map defined by putting

$$\begin{aligned} \Lambda_k^*(\sigma, u_2, \ldots, u_n, \delta) &= F_k(a_1, u_2, \ldots, u_n, \sigma) - u_k, \qquad k = 3, \ldots, n, \\ \Lambda_1^*(\sigma, u_2, \ldots, u_n, \delta) &= F_1(a_1, u_2, \ldots, u_n, \sigma) - a_1, \\ \Lambda_2^*(\sigma, u_2, \ldots, u_n, \delta) &= \eta(a_1, u_2, \ldots, u_n) - \delta \end{aligned} \tag{6}$$

where $\eta: U \to \mathbb{R}$ is the integral for (1).

One of the solutions of this system is $(\omega, a_2, \dots, a_n, \Delta)$, which corresponds to the periodic solution ϕ_0 of (1) with period ω, initial condition $\phi_0(0) = (a_1, \dots, a_n)$, and $\eta(a) = \Delta$.

The matrix of the derivative of the map $(\sigma, u_2, \dots, u_n) \mapsto \Lambda^*(\sigma, u_2, \dots, u_n, \delta)$ at $(\omega, a_2, \dots, a_n)$ is

$$\begin{pmatrix} X_1(a) & 0 & \\ 0 & D_2\eta(a) & \mathbf{0} \\ & \mathbf{0} & M' \end{pmatrix}.$$

This matrix is nonsingular as $X_1(a)$ and $D_2\eta(a)$ are nonzero and M' is an $(n-2) \times (n-2)$ nonsingular matrix. Therefore, from the implicit function theorem, there exist $d > 0$ and $s > 0$ and a differentiable function g: $(\Delta - d, \Delta + d) \rightarrow W \subset J \times A$ such that $g(\Delta) = (\omega, a_2, \dots, a_n)$ and for all δ in the domain of g

(i) $\Lambda^*(g(\delta), \delta) = 0$;

(ii) $g(\delta)$ is the only solution of Eq. (i) which lies in the ball $B_s(\omega, a_2, \dots, a_n)$.

We now define a map f: $(\Delta - d, \Delta + d) \rightarrow \{a_1\} \times A \subset V$ by putting

$$f(\delta) = (a_1, g_2(\delta), \dots, g_n(\delta)).$$

Thus f maps the interval $(\Delta - d, \Delta + d)$ onto its image, denoted by $\mathscr{I}$, which consists of initial conditions for periodic solutions of the differential equation (1). A period of $f(\delta)$ is $g_1(\delta)$.

To prove that f: $(\Delta - d, \Delta + d) \rightarrow \mathscr{I}$ *is a diffeomorphism.* We prove first that f is injective. Suppose that $f(\beta) = f(\delta)$ so that by the definition of f

$$\Lambda^*(g_1(\beta), f_2(\beta), \dots, f_n(\beta), \beta) = 0 = \Lambda^*(g_1(\delta), f_2(\delta), \dots, f_n(\delta), \delta).$$

It now follows from (6) that $\beta = \delta$, each of these numbers being the value of the integral η: $U \rightarrow \mathbb{R}$ at a single point of its domain. Thus the map f: $(\Delta - d, \Delta + d) \rightarrow \mathscr{I}$ is injective.

The inverse of f is the map

$$(a_1, u_2, \dots, u_n) \mapsto \delta = \eta(a_1, u_2, \dots, u_n)$$

restricted to $\mathscr{I}$ as domain. This is just the map $\eta \mid \mathscr{I}$ where η: $U \rightarrow \mathbb{R}$ is the integral for (1). By assumption the integral is differentiable on U and so the inverse of f is differentiable. Hence f is a diffeomorphism.

A family of periodic solutions Ψ: $(\Delta - d, \Delta + d) \rightarrow \mathscr{B}^r(L, \mathbb{R}^n)$ *of* (1). To verify that $\mathscr{P}$ is the image of a family of periodic solutions to (1) we must prove that the prime period function is bounded and continuous on $(\Delta - d, \Delta + d)$ and then use Lemma 5.4 to complete the proof.

The map $g\colon (\Delta - d, \Delta + d) \to W \subset J \times A$ is continuous and so its first component $g_1 : \delta \to \sigma$ is continuous, where σ is the period of the solution with initial condition $(a_1, g_2(\delta),..., g_n(\delta))$. The function g_1 is the prime period function (after possible restriction to a smaller open interval as domain). Otherwise there would be a sequence $\{\gamma_n\}_{n=1}^{\infty}$ with limit Δ such that for each n, $g_1(\delta)$ is not the prime period of the solution $\Psi(\gamma_n)$. Therefore, the prime period of $\Psi(\gamma_n)$ is one of the numbers $\frac{1}{2} g_1(\gamma_n)$, $\frac{1}{3} g_1(\gamma_n)$,.... But this contradicts Lemma 5.2 with $\omega^* = \frac{3}{4}\omega$. Therefore the prime period of each member of the family is given by the function g_1 and so the prime period function is continuous.

Further, the prime period function is bounded. This follows from (ii), which gives us directly that $g_1(\delta) < \omega + s$ for each $\delta \in (\Delta - d, \Delta + d)$.

Lemma 5.4 now tells us that the set of periodic solutions to (1), $\mathscr{P}$, with initial conditions given by the diffeomorphism f of the interval $(\Delta - d, \Delta + d)$, is the image of a family of periodic solutions $\Psi\colon (\Delta - d, \Delta + d) \to \mathscr{B}^r(L, \mathbb{R}^n)$.

Proof that Ψ is a proper family. We wish to prove that each reparametrization of Ψ is nowhere tangent to a trivial family. For any periodic solution ϕ of (1) the trivial family $\Phi\colon \mathbb{R} \to \mathscr{B}^r(L, \mathbb{R}^n)$ is defined by

$$\Phi(\gamma) = \phi \circ (\iota + \gamma)$$

where $\iota\colon \mathbb{R} \to \mathbb{R}$ is the identity function. The derivative of this family at each $\gamma \in \mathbb{R}$ is

$$\Phi'(\gamma) = \phi' \circ (\iota + \gamma).$$

Now we have assumed that $(\phi_0')_1(0) = X_1(a) \neq 0$, so by choosing d sufficiently small we may assume: For any periodic solution $\phi\colon L \to \mathbb{R}$ of (1) which is a member of the family $\Psi\colon (\Delta - d, \Delta + d) \to \mathscr{B}^r(L, \mathbb{R}^n)$, $\phi_1'(0) \neq 0$. This implies that for the trivial family Φ corresponding to ϕ,

$$\Phi_1'(0)(0) = \phi_1'(0) \neq 0.$$

But we prove that for the family we have constructed

$$\Psi_1'(0)(0) = 0.$$

This shows that $\Psi\colon (\Delta - d, \Delta + d) \to \mathscr{B}^r(L, \mathbb{R}^n)$ is nowhere tangent to a trivial family and so is a proper family of periodic solutions to (1).

To check $\Psi_1'(0)(0) = 0$, note that by the definition of Fréchet differentiation the derivative of the function $G\colon \gamma \mapsto \Psi_1(\gamma)(0)$ is equal to $\Psi_1'(\gamma)(0)$. For each $\gamma \in (\Delta - d, \Delta + d)$,

$$G(\gamma) = a_1,$$

as we choose the initial conditions used in the implicit function theorem to have first component equal to a_1. Therefore $G'(0) = 0$, and so

$$\Psi_1'(0)(0) = 0.$$

Proof that Ψ satisfies (a) *and* (b) *of the embedding theorem.* From Lemma 5.4 we know that Ψ: $(\Delta - d, \Delta + d) \to \mathscr{P}$ maps each $\delta \in (\Delta - d, \Delta + d)$ onto the periodic solution with initial condition $f(\delta)$. Therefore,

$$\Psi(\Delta) = \phi_0 ,$$

the periodic solution with initial condition $f(\Delta) = a$.

To prove (b), proceed by contradiction. Suppose that for each $s' > 0$ there is a periodic solution to (1), ϕ: $\mathbb{R} \to U$ with prime period $\theta \in (\omega - s', \omega + s')$ such that $\phi \in B_{s'}(\phi_0)$ and $\phi \notin \Psi[(\Delta - d, \Delta + d)]$. Hence

$$|\phi(0) - \phi_0(0)| < s'.$$

Choose $s' < s$ such that $\phi(0) \in N$, the neighborhood of $\phi_0(0)$ given by Lemma 5.1, so that the solution ϕ passes through the hyperplane $u_1 = a_1$. Hence $\phi(0) \in f[(\Delta - d, \Delta + d)]$. But from Lemma 5.1, and the result (ii) of the use of the implicit function theorem above, and the definition of f, f: $(\Delta - d, \Delta + d) \to \mathscr{I}$ gives the initial conditions of all periodic solutions to (1) contained in the open ball $B_s(\omega, a_2, \ldots, a_n)$. Through each point of U there is only one solution of (1), and so the period of ϕ is given by the function $g_1 : (\Delta - d, \Delta + d) \to (\omega - s, \omega + s)$ which was shown above to be the prime period function. Therefore there exists δ such that $g_1(\delta) = \theta$ and $f(\delta) = (a_1, \phi_2(0), \ldots, \phi_n(0))$, and from the definition of f, $g(\delta) = (\theta, \phi_2(0), \ldots, \phi_n(0))$, so that $\phi \in \Psi[(\Delta - d, \Delta + d)]$, contradicting our hypothesis, and so (b) is proved.

To complete this section we prove Lemma 5.4.

Proof of Lemma 5.4. The evaluation map ev: $\mathscr{B}^1(L, \mathbb{R}^n) \times L \to \mathbb{R}^n$ is defined by

$$\mathrm{ev}(g, t) = g(t)$$

for $g \in \mathscr{B}^1(L, \mathbb{R}^n)$ and $t \in L$. Abraham and Robbin [2, p. 25] prove that this map is differentiable on $\mathscr{B}^1(L, \mathbb{R}^n) \times \mathbb{R}$. Therefore the restricted map $\mathrm{ev} \mid \mathscr{P} \times \{0\}$: $\phi_\alpha \to \phi_\alpha(0)$ is differentiable where ϕ_α denotes the member of $\mathscr{P}$ with initial condition $\phi_\alpha(0) = \alpha \in \mathscr{I}$ so that this function maps $\mathscr{P}$ injectively onto $\mathscr{I}$. As f: $I \to \mathscr{I}$ is a diffeomorphism of f onto $\mathscr{I}$ then $f^{-1} \circ \mathrm{ev} \mid \mathscr{P} \times \{0\}$ maps $\mathscr{P}$ differentiably and injectively onto $\mathscr{I}$.

To complete the proof of this lemma we must prove that the inverse of this map is differentiable. For each $\gamma \in I$ the inverse is Ψ: $I \to \mathscr{P}$, where

$$\Psi(\gamma) = (t \mapsto F(f(\gamma), t))$$

and F: $V \times J \to U$ is a flow box for (1), U having a nonempty intersection with $\mathscr{I}$.

As $\mathbb{R}^n$ is locally compact we may choose a compact neighborhood $N(\gamma)$ of some $f(\gamma)$ in V such that $N(\gamma) \subset V$. Further, we assumed that the prime period function was bounded, and so, if Ω is a bound then we may assume that each periodic solution in $\mathscr{P}$ has domain $L = [0, \Omega]$. Therefore the derivative of the function $G: (\gamma', t') \mapsto F(f(\gamma'), t')$ is defined on the compact set $f^{-1}[N(\gamma)] \times L$ and so is uniformly continuous on $f^{-1}[N(\gamma)] \times L$. From [4, p. 159, Problem 36] we see that for each $\epsilon > 0$ there is $\delta > 0$ such that for $(\beta, s) \in f^{-1}[N(\gamma)] \times L$,

$$|(\beta, s)| < \delta \quad \text{and} \quad (\gamma, t) + \theta(\beta, s) \in f^{-1}[N(\gamma)] \times L, \quad \text{for} \quad \theta \in [0, 1],$$

ensures that

$$|F(f(\gamma + \beta), t + s) - F(f(\gamma), t) - DG(\gamma, t)(\beta, s)| \leqslant \epsilon\, |(\beta, s)|.$$

If we let $s = 0$ and take the supremum with respect to t over $[0, \Omega]$ we get, where $\Psi(\gamma) = (t \mapsto F(f(\gamma), t))$,

$$\| \Psi(\gamma + \beta) - \Psi(\gamma) - M(\gamma)(\beta)\| \leqslant \epsilon\, |(\beta, 0)|,$$

where $M(\gamma) = (\gamma \mapsto DG(\gamma, t)(\beta, 0))$

Hence Ψ is differentiable on I.

6. Proof of the Extension Theorem

To prove the extension theorem we need a geometrical lemma whose proof is motivated by the techniques used by Milnor [6, pp. 55–57] in classifying one-dimensional manifolds.

LEMMA 6.1. *Let $f: I = (a, b) \to V$ and g: $J = (c - d, c + d) \to V$ be diffeomorphisms of the bounded intervals I and J into a Banach space V. If there is an open ball B such that*

(i) $g[J] \subset B$ *and* $f[I] \cap B \subseteq g[J]$,

then

(I) $f[I] \cap g[J]$ *has at most two connected components.*

If, furthermore,

(ii) *there is a sequence* $\{\gamma_n\}_{n=1}^{\infty}$ *in I such that*

$$\gamma_n \to b \quad \textit{and} \quad f(\gamma_n) \to g(c) \quad \textit{as} \quad n \to \infty,$$

then either

(IIa) $f[I] \cap g[J]$ *has one component and some reparametrization of f can be extended to a diffeomorphism, f^* onto $f[I] \cup g[J]$, which properly contains $f[I]$, or,* (IIb) *$f[I] \cap g[J]$ has two components and $f[I] \cup g[J]$ is diffeomorphic to S^1, the unit circle.*

Note. For later reference we need to know that the extended diffeomorphism $f^*: (a', c + d) \to V$ referred to in (IIa) is given by

$$\text{(III)} \quad \begin{aligned} f^*(\gamma) &= (f \circ M)(\gamma), && \gamma \in (a', b'), \\ &= (f \circ (f^{-1} \circ g))(\gamma), && \gamma \in [b', c), \\ &= g(\gamma), && \gamma \in [c, c + d), \end{aligned}$$

where $M: \mathbb{R} \to \mathbb{R}$ is linear with slope $(f^{-1} \circ g)'(b')$, $M(a') = a$, and where $f[(\beta, b)]$ is the connected component of $f[I] \cap g[J]$, $b' \in g^{-1}[f[(\beta, b)]]$.

Proof of Lemma 6.1, *Part* (I). Clearly (I) follows if we can show that $f^{-1} \circ g$ maps the disjoint union of at most two open intervals onto another such set.

To see this note first that $f^{-1} \circ g: K \to I$, where $K = (g^{-1} \circ f)[I \cap f^{-1}[B]]$. But by hypothesis (i),

$$\begin{aligned} f^{-1}(g[K]) &= f^{-1}[f[I] \cap B] \\ &= I \cap f^{-1}[B], \end{aligned}$$

which is open in I. Hence K is open. Clearly $f^{-1} \circ g$, is a diffeomorphism of an open subset of J onto an open subset of I.

Second, consider the set $\Gamma \subseteq I \times J$ consisting of all (s, t) with $f(s) = g(t)$. Since every open set on the real line is a disjoint union of open intervals Γ will consist of a disjoint union of curve segments (see Fig. 1).

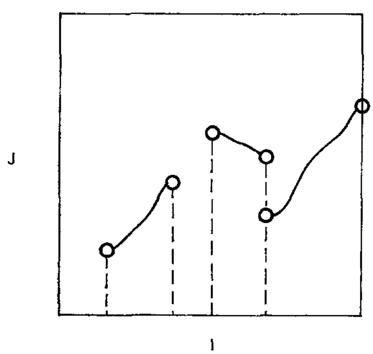

FIGURE 1

Each of these segments is diffeomorphic to an open interval, while Γ is closed in $I \times J$. Hence the end points of these segments must lie on the boundary of $I \times J$. Since f and g are functions, horizontal and vertical lines can cross only once, and since $f^{-1} \circ g$ is a diffeomorphism each segment has slope of constant sign. These constraints lead to pictures for Γ of which those shown in Fig. 2 are typical. In all cases one finds that both the domain and range of $f^{-1} \circ g$ has at most two connected components. This completes the proof of (I).

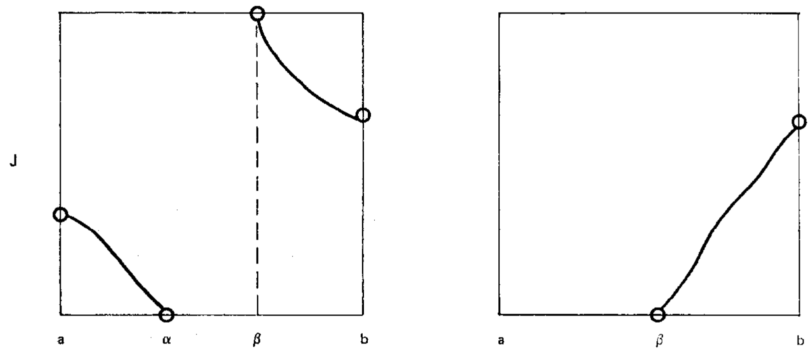

FIGURE 2

Proof of (II). Suppose now that (ii) holds so that the set Γ comes arbitrarily close to the point (b, c) (see Fig. 3). Now suppose that $f[I] \cap g[J]$ has one component. We may assume without loss of generality that $(f^{-1} \circ g)' > 0$ and so from the second part of (i) and the graph (Fig. 3) we see that $(f^{-1} \circ g)$ maps the interval $(c - d, c)$ diffeomorphically onto (β, b), where $\beta \in (a, b)$. We may use formula (III) above to define a map f^*. It is clear from (III) and (i) that f^* is one to one. Further, from (III), we see that at each point of its domain f^* is differentiable with differentiable inverse. Therefore f^* is a diffeomorphism onto $f[I] \cup g[J]$, which properly contains $f[I]$.

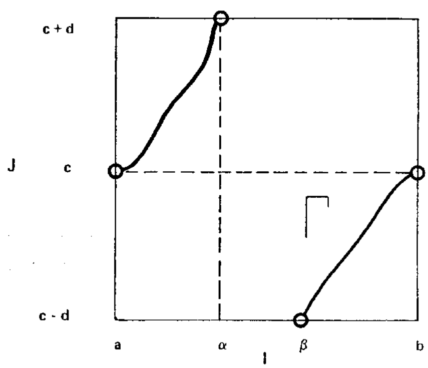

FIGURE 3

Finally suppose that $f[I] \cap g[J]$ has two components. An argument analogous to that given by Milnor [6, p. 56] establishes conclusion (IIb) and thereby completes the proof of the lemma.

To prove Proposition 3.1 we use the following lemma.

LEMMA 6.2. *Let* Φ: $(a, b) \to \mathscr{B}^r(L, \mathbb{R}^n)$ *be a family of periodic solutions to* (1). *If* X: $\mathrm{cl}(U) \subseteq \mathbb{R}^n \to \mathbb{R}^n$, *the right-hand side of the autonomous differential equation* (1), *is* $\mathscr{C}^2$ *then for each* $\beta, \gamma \in (a, b)$ *and for all* $t \in L$

$$|\Phi(\gamma)(t) - \Phi(\beta)(t)| \leqslant |\Phi(\gamma)(0) - \Phi(\beta)(0)| \exp(At),$$

where A *is the Lipschitz constant for* X.

The proof of this lemma is given in [5, p. 380].

Let us recall some notation used in Section 3. Let Φ: $(a, b) \to \mathscr{B}^r(L, \mathbb{R}^n)$ be a family of periodic solutions to (1) and let η: $U \subseteq \mathbb{R}^n \to \mathbb{R}$ be an integral for (1). We defined two maps τ, s: $(a, b) \to \mathbb{R}^+$ by

$$\tau(\gamma) = \text{prime period of the periodic solution } \Phi(\gamma) \text{ of } (1)$$

and

$$s(\gamma) = \mathrm{Sup}\{|\Phi(\gamma)(t)| : t \in L\}$$

for each $\gamma \in (a, b)$.

Proof of Proposition 3.1. Here we assume that

$$\tau, s\colon (a, b) \to \mathbb{R}^+ \text{ are bounded.}$$

Then there is a sequence $\{\beta_m\}_{m=1}^{\infty}$ in (a, b) such that $\beta_m \to b$ as $m \to \infty$ and $\lim_{m\to\infty} \tau(\beta_m)$ exists. Further, as s is bounded the set $\{\Phi(\beta_m)(0): \beta_m \in (a, b)\}$ is a bounded subset of $\mathrm{cl}(U)$. Therefore we can select from the sequence $\{\beta_m\}_{m=1}^{\infty}$ a subsequence, $\{\gamma_n\}_{n=1}^{\infty}$, such that $\gamma_n \to b$ as $n \to \infty$ and $\lim_{n\to\infty} \Phi(\gamma_n)(0)$ exists and lies in $\mathrm{cl}(U)$.

Construction of a solution ϕ_0 *to* (1). For each pair of integers p and q

$$\Phi(\gamma_p)' = X \circ \Phi(\gamma_p)$$

and

$$\Phi(\gamma_q)' = X \circ \Phi(\gamma_q)$$

as Φ: $(a, b) \to \mathscr{B}^r(L, \mathbb{R}^n)$ is a family of periodic solutions to (1). Because s and τ are bounded there is a closed cube containing the images of all the (periodic) solutions of the family Φ. As the vector field X is $\mathscr{C}^2$ on $\mathrm{cl}(U)$ intersected with this cube there is a constant A such that

$$\begin{aligned}\|\Phi(\gamma_p)' - \Phi(\gamma_q)'\| &= \mathrm{Sup}\{|X(\Phi(\gamma_p)(t)) - X(\Phi(\gamma_q)(t))| : t \in L\}\\ &\leqslant A\,\mathrm{Sup}\{|\Phi(\gamma_p)(t) - \Phi(\gamma_q)(t)| : t \in L\}.\end{aligned}$$

As the prime period function τ: $(a, b) \to \mathbb{R}^+$ is bounded there exists $\Omega \in \mathbb{R}$ such that for each $\gamma \in (a, b)$ the prime period of the solution $\Phi(\gamma)$ satisfies $\tau(\gamma) \leqslant \Omega$.

We may assume that each periodic solution of the family Φ has domain $L = [0, \Omega]$. From Lemma 6.2 we see that

$$\begin{aligned}\| \Phi(\gamma_p)' - \Phi(\gamma_q)' \| &\leqslant A \operatorname{Sup}\{| \Phi(\gamma_p)(0) - \Phi(\gamma_q)(0)| \exp At : t \in [0, \Omega]\} \\ &= A \,| \Phi(\gamma_p)(0) - \Phi(\gamma_q)(0)| \exp A\Omega.\end{aligned}$$

But the sequence $\{\Phi(\gamma_p)(0)\}_{p=1}^{\infty}$ is convergent so that the sequence $\{\Phi(\gamma_p)'\}_{p=1}^{\infty}$ is a Cauchy sequence. $\mathscr{B}^r(L, \mathrm{cl}(U))$ is complete so this sequence has a limit in $\mathscr{B}^r(L, \mathrm{cl}(U))$. Thus since the convergence of the derivatives is uniform,

$$\lim_{n\to\infty} \Phi(\gamma_n)' = (\lim_{n\to\infty} \Phi(\gamma_n))'.$$

But for $t \in [0, \Omega]$

$$\lim_{n\to\infty} \Phi(\gamma_n)'(t) = \lim_{n\to\infty}(X \circ \Phi(\gamma_n))(t) = (X \circ \lim_{n\to\infty} \Phi(\gamma_n))(t)$$

as X is continuous on $\mathrm{cl}(U)$. Therefore

$$\lim_{n\to\infty} \Phi(\gamma_n)' = X \circ (\lim_{n\to\infty} \Phi(\gamma_n));$$

that is, the function

$$\phi_0 = \lim \Phi(\gamma_n)$$

satisfies the differential equation (1), although ϕ_0 may assume values on the boundary of U.

To Prove ϕ_0 is periodic. For each n

$$\Phi(\gamma_n)(0) = \Phi(\gamma_n)(\tau(\gamma_n))$$

So if we prove that

$$\lim_{n\to\infty} \Phi(\gamma_n)(\tau(\gamma_n)) = \phi_0(\lim_{n\to\infty} \tau(\gamma_n)), \tag{7}$$

it will follow that

$$\phi_0(0) = \phi_0(\omega),$$

where $\omega = \lim_{n\to\infty} \tau(\gamma_n)$. Therefore ϕ_0 will be a periodic solution, unless $\omega = 0$, in which case it will be an equilibrium solution.

So it remains to prove (7):

$$\begin{aligned}&| \Phi(\gamma_n)(\tau(\gamma_n)) - \phi_0(\omega)| \\ &\qquad\leqslant | \Phi(\gamma_n)(\tau(\gamma_n)) - \phi_0(\tau(\gamma_n))| + | \phi_0(\tau(\gamma_n)) - \phi_0(\omega)| \\ &\qquad\leqslant | \Phi(\gamma_n)(0) - \phi_0(0)| \exp A\omega + | \phi_0(\tau(\gamma_n)) - \phi_0(\omega)|\end{aligned}$$

(from Lemma 6.2). $\Phi(\gamma_n)$ converges uniformly to ϕ_0 so that the first term is small for large n. The solution ϕ_0 is continuous and so the second term is small for large n. Therefore (7) is proved.

Proof of the extension theorem. Suppose that τ, s: $(a, b) \to \mathbb{R}^+$ are bounded functions. Then from Proposition 3.1 there is a sequence $\{\gamma_n\}_{n=1}^{\infty}$ in (a, b) converging to b and a solution ϕ_0 of (1) such that

$$\Phi(\gamma_n) \to \phi_0 \qquad \text{as} \qquad n \to \infty.$$

Further, in accordance with hypotheses (B), (C), and (D) of the extension theorem assume that

(i) ϕ_0 is a periodic solution of (1) with prime period $\omega > 0$ (that is, ϕ_0 is not an equilibrium solution) which is equal to

$$\lim_{\gamma \to b^-} \tau(\gamma),$$

(ii) the characteristic multiplier 1 of (1) ϕ_0 has multiplicity at most 2, and

(iii) the integral η: $U \to \mathbb{R}$ of (1) is nonstationary on $\phi_0[L]$.

Conditions (i), (ii), and (iii) are the hypotheses of the embedding theorem (Theorem 3.4) so that there is a proper family of periodic solutions Ψ: $(c - d, c + d) \to \mathscr{P} \subset \mathscr{B}^r(L, \mathbb{R}^n)$ to (1) such that

(a) $\Psi(c) = \phi_0$,

(b) there exists $\rho > 0$ such that $\mathscr{P}$ contains all periodic solutions of (1) which lie in the open ball $B_\rho(\phi_0) \subset \mathscr{B}^r(L, \mathbb{R}^n)$ and have prime period in the interval $(\omega - \rho, \omega + \rho)$.

If in Lemma 6.1 we let $g = \Psi$, $f = \Phi$, $g(c) = \phi_0$, and let $B = B_\rho(\phi_0)$ then $\Phi[(a, b)] \cap \Psi[(c - d, c + d)]$ has at most two components. Further, either $\Phi[(a, b)] \cup \Psi[(c - d, c + d)]$ is diffeomorphic to S^1, which completes the proof, or there is a reparametrization of Φ which can be extended to a diffeomorphism Φ^*: $K \to \Phi[(a, b)] \cup \Psi[(c - d, c + d)]$ for some interval K using f^* of Lemma 6.1. From the embedding theorem, Theorem 5.5, the prime period is continuous along the family. Therefore Φ^* is a family which is an extension on the right of the family Φ.

To complete the proof we must show that if Φ is a proper family of periodic solutions to (1) then so is Φ^*. From the note to Lemma 6.2 we see that

$$\begin{aligned} \Phi^{*\prime}(\gamma) &= \Delta \cdot \Phi'(L(\gamma)), && \gamma \in (a', b'), \\ &= \Psi'(\gamma), && \gamma \in [b', b + c), \end{aligned} \tag{8}$$

where $\Delta = M'(\gamma) = (\Phi^{-1} \circ \Psi)'(b')$. That the family Ψ is proper follows from the embedding theorem and so both Φ and Ψ are proper families. That is, each reparametrization of Φ and Ψ is nowhere tangent to a trivial family. In view of (8), Φ^* is nowhere tangent to a trivial family and Φ^* is a proper right extension of Φ.

Acknowledgment

The author acknowledges gratefully the assistance of Dr A. R. Jones, who freely shared his knowledge of this subject with the author during the preparation of this work.

References

1. R. Abraham and J. Marsden, "Foundations of Mechanics," Benjamin, New York, 1967.
2. R. Abraham and J. Robbin, "Transversal Mappings and Flows," Benjamin, New York, 1967.
3. G. D. Birkhoff, "Collected Mathematical Papers," Vol. 2, pp. 706–709, Dover, New York, 1968.
4. J. Dieudonné, "Foundations of Modern Analysis," Academic Press, New York, 1960.
5. S. Lang, "Analysis," Vol. I, Addison–Wesley, Reading, Mass., 1968.
6. J. Milnor, "Topology from the Differentiable Viewpoint," Univ. Press of Virginia, Charlottesville, 1969.
7. C. Siegel and J. Moser, "Lectures on Celestial Mechanics," Springer-Verlag, Berlin, 1971.
8. V. Szebehly, "Theory of Orbits," Academic Press, New York, 1967.
9. A. Wintner, "The Analytical Foundations of Celestial Mechanics," Princeton Univ. Press, Princeton, N.J., 1941.
10. A. Wintner, Grundlagen einer Genealogie der periodischen Bahnen im restringierten Dreikörper problem, *Math. Z.* **34** (1932), 321–349.
11. A. Wintner, Three notes on characteristic exponents and equations of variation in celestial mechanics, *Amer. J. Math.* **53** (1931), 605–625.